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A FORMULATION OF INELASTICITY FROM THERMAL AND MECHANICAL CONCEPTS

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Abstract—Technical problems involving inelastic materials are mathematically nonlinear and mechanically non-conservative. Consequently, the underlying theories must be founded upon the concepts of thermodynamics of incremental deformations [1], and the methods of solutions are invariably accomplished by numerical procedures which are achieved by solving a succession of linear equations governing the incremental deformations.

Thermodynamic criteria provide the bases of variational theorems for inelastic bodies. The two presented here, are extensions of the Hu-Washizu and the Hellinger-Reissner theorems for elastic bodies.

The formulation of the viscous-plastic-elastic theory is based upon thermomechanical concepts and developed in an incremental form suited to practical computations. Incremental forms of the free energy and dissipation are expressed in terms of the strains, temperature and inelastic strains. These forms are related to specific mechanical models and all quantities are identified with familiar geometrical, mechanical and thermal variables. In particular, terms of the free energy potential depend upon elastic properties while the terms of the dissipation function involve the inelastic (viscous and plastic) properties.

The development is motivated by earlier works [2, 3] on plasticity without a yield surface. Indeed, certain forms are contained in Valanis' work [2]. However, specific forms are correlated with classical theories of work-hardening plasticity, but essential differences are also noted.

NOTATION

s	surface of undeformed body
v	volume of undeformed body
E, P	sufficies denote elastic, plastic
N	suffix denotes an inelastic strain and the conjugate dissipative stress ($N = 1, \ldots, M$)
θ^i	spatial coordinate
$\dot{}$	overdot signifies material time derivative
$'$	overprime signifies an increment
—	underline signifies a prescribed value; overbar signifies the reference state
$,_i$	coma signifies spatial derivative $,_i \equiv (\partial/\partial\theta^i)$
$\mathbf{r}, \mathbf{R}$	position vector of initial, current state
$\mathbf{V}$	displacement: $\mathbf{V} = \mathbf{R} - \mathbf{r}$
$\mathbf{g}_i, \mathbf{G}_i$	tangent vector of initial, current state: $\mathbf{g}_i = \mathbf{r}_{,i}$, $\mathbf{G}_i = \mathbf{R}_{,i}$
$\mathbf{g}^i, \mathbf{G}^i$	reciprocal vectors: $\mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j$
g_{ij}, G_{ij}	covariant component of metric tensor $g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j$, $G_{ij} \equiv \mathbf{G}_i \cdot \mathbf{G}_j$
$\sqrt{g}, \sqrt{G}$	metric of initial, current volume: $\sqrt{(g)} = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)$, $\sqrt{(G)} = \mathbf{G}_1 \cdot (\mathbf{G}_2 \times \mathbf{G}_3)$
γ_{ij}	component of strain tensor $\gamma_{ij} \equiv \frac{1}{2}(G_{ij} - g_{ij})$
s^{ij}	component of symmetric (Piola-Kirchhoff) stress tensor $s^{ij} = \mathbf{t}^i \cdot \mathbf{G}^j \sqrt{(g^{(ii)})}$
$\mathbf{t}^i$	stress vector (force/undeformed area) upon θ^i surface
U	internal energy†
S	entropy†
Q	heat†
$\dot{\mathbf{q}}$	heat flux (rate per unit initial area)
$\dot{h}$	internal heat rate†
D	energy dissipated†
T	absolute temperature
ρ	mass density of initial state
F	free energy†
G	Gibbs potential†
H	enthalpy†
f	yield function
$\alpha \equiv (E_1/\lambda)$	dimensionless parameter of plastoelasticity
β	dimensionless hardening parameter
$\varphi \equiv (E_0/E_1)$	dimensionless parameter of model 2
$\eta = (E_1/\mu)$	dimensionless parameter of viscoelasticity

†Per unit initial volume.

INTRODUCTION

The classical theory of plasticity has existed for decades in a generally acceptable form. Yet few formal solutions have been obtained for the numerous problems of technological importance and, only recently, some numerical results have been obtained for simple problems of two and three dimensions.

Certain difficulties are inherent in problems of inelastic bodies: the problems are mathematically *nonlinear*. Moreover, the stresses are *nonconservative* so that a solution is path dependent; in practice, an approximation is constructed from successive, piece-wise linear, "solutions". Still another difficulty stems from the concept of a yield condition which signals an abrupt transition from elastic to inelastic behavior. Mathematically, the yield condition introduces distinct regions, elastic and inelastic, which are governed by *different* constitutive equations and which change as the deformation progresses.

A theory without the yield condition has been advanced by Valanis[2]. His "endochronic" theory is attractive for several reasons: It eliminates the abrupt transition from elastic to inelastic states and it avoids the attendant mathematical problems: It is drawn from thermal-mechanical concepts and also overcomes some physical shortcomings of the classical theory.

Here, constitutive equations of endochronic theories are also formulated in concert with classical thermodynamics. The equations of plasticity are the equations of Valanis' theory. However, the formulation and interpretation have some novel and useful features.

Firstly, the thermodynamic conditions are embodied in two variational theorems which extend the theorem of Hu-Washizu and the theorem of Hellinger-Reissner, respectively. The theorems provide bases for constructing approximations, for example, the discrete systems of finite elements.

Various forms of the free energy and dissipation are drawn from specific mechanical models and the constitutive equations are derived in accordance with the thermodynamic criteria. Consequently, all coefficients of the energy potentials are identified with elastic properties while coefficients of the dissipation are identified with viscous or plastic properties. Other equations, which describe various attributes of viscous-elastic-plastic solids, are readily constructed, as an assembly of mechanical elements.

An endochronic theory, which admits additivity of elastic and plastic strain rates (a premise of classical plasticity), is placed within the context of the classical theory. An endochronic theory, which exhibits the Bauschinger effect, is compared with the classical theory of kinematic hardening.

THERMODYNAMICAL CONCEPTS

The first law of thermodynamics asserts a balance of energy:

$$-\iint_s \dot{q}^i n_i \,\mathrm{d}s + \iiint_v \dot{h}\,\mathrm{d}v + \iint_{s_1} \mathbf{t}\cdot\dot{\mathbf{V}}\,\mathrm{d}s + \iiint_v \rho\mathbf{f}\cdot\dot{\mathbf{V}}\,\mathrm{d}v$$

$$= \iiint_v \left[\dot{U} + \frac{1}{2}\rho(\dot{\mathbf{V}}\cdot\dot{\mathbf{V}})\right]\mathrm{d}v. \tag{1a}$$

Here the integrals on the left account for the rates of heat flow (flux $\dot{\mathbf{q}}$) through the surface (s), heat generated (h) within the body (v), work done by tractions ($\mathbf{t}$) upon the surface (s_1), and the work done by body force ($\mathbf{f}$) within the body (v). The terms on the right represent rates of the internal energy (U) and kinetic energy. Integrating by parts we obtain the volume integral:

$$\iiint_v [\dot{Q} + s^{ij}\dot{\gamma}_{ij} - \dot{U}]\,\mathrm{d}v = 0. \tag{1b}$$

Here, we assume that the stress satisfies the equations of motion and denote the specific heat rate by

$$\dot{Q} = -\frac{1}{\sqrt{g}}(\dot{q}^i\sqrt{g})_{,i} + \dot{h}.$$

The notions of reversibility and irreversibility are related to the entropy function S. If a process is reversible, then the entropy has the differential:

$$dS \equiv \frac{dQ}{T} \tag{2a}$$

where T denotes the absolute temperature. Stated otherwise,

$$\dot{Q} = T\dot{S} \tag{2b}$$

To express the second law in a meaningful way we *define* a dissipation function D, such that

$$\dot{D} \equiv T\dot{S} - \dot{Q}. \tag{3}$$

Then the second law of thermodynamics asserts that

$$\dot{D} \geq 0. \tag{4}$$

The equality holds for a reversible process and the inequality for an irreversible process.
Substituting (3) into (1b), we obtain

$$\iiint_v [s^{ij}\dot{\gamma}_{ij} + T\dot{S} - \dot{U} - \dot{D}]dv = 0. \tag{5a}$$

Since (5a) holds for any part of the body, the integrand must vanish.

$$s^{ij}\dot{\gamma}_{ij} + T\dot{S} - \dot{U} = \dot{D} \geq 0. \tag{5b}$$

The initial term of (5) is the rate of work expended by the stresses and the second is the rate of heat supplied in a reversible process. Accordingly, (5) states that the combined rates of work and heat equal the rate of internal energy if the process is reversible (=) and exceed the rate of internal energy if the process is irreversible (>).

ELASTIC DEFORMATIONS (REVERSIBLE PROCESSES)

We postulate that the thermodynamical state of a simple elastic material is determined by the strain components and the absolute temperature. We further accept the elastic deformation as a reversible process. Then,

$$S = S(\gamma_{ij}, T). \tag{6a}$$

Assuming that the function has continuous nonvanishing derivatives, we may write

$$T = T(\gamma_{ij}, S). \tag{6b}$$

Stated otherwise, S and T are alternative independent variables. Therefore, the internal energy may be regarded as a function of the strain components and the entropy:

$$U = U(\gamma_{ij}, S). \tag{7}$$

According to the equality (5) and (7), we have

$$\left(s^{ij} - \frac{\partial U}{\partial \gamma_{ij}}\right)\dot{\gamma}_{ij} + \left(T - \frac{\partial U}{\partial S}\right)\dot{S} = 0. \tag{8}$$

Since γ_{ij} and S are independent, it follows that

$$S^{ij} = \frac{\partial U}{\partial \gamma_{ij}}, \qquad T = \frac{\partial U}{\partial S}. \tag{9a, b}$$

Equations (9a, b) admit the possibility of a Legendre transformation [4] wherein the variable s^{ij} is an alternative to γ_{ij} as S is an alternative to T. Accordingly, one defines the *free energy*:

$$F(\gamma_{ij}, T) \equiv U - ST. \tag{10}$$

By forming the time-derivative as in (8) we obtain

$$s^{ij} = \frac{\partial F}{\partial \gamma_{ij}}, \qquad S = -\frac{\partial F}{\partial T}, \tag{11}$$

The Gibbs potential G and enthalpy H are likewise defined by transformations, as follow:

$$G(s^{ij}, T) = U - s^{ij}\gamma_{ij} - TS \tag{12}$$

$$\gamma_{ij} = -\frac{\partial G}{\partial s^{ij}}, \qquad S = -\frac{\partial G}{\partial T} \tag{13a, b}$$

$$H(s^{ij}, S) = U - s^{ij}\gamma_{ij} \tag{14}$$

$$\gamma_{ij} = -\frac{\partial H}{\partial s^{ij}}, \qquad T = \frac{\partial H}{\partial S}. \tag{15a, b}$$

In an adiabatic deformation ($\dot{S} = 0$), $U(\gamma_{ij})$ and $H(s^{ij})$ are the so-called complementary potentials of elasticity. In an isothermal deformation ($\dot{T} = 0$), $F(\gamma_{ij})$ and $G(s^{ij})$ are the corresponding complementary potentials.

INELASTIC DEFORMATION (IRREVERSIBLE PROCESSES)

The occurrence of inelastic strains γ_{ij}^N ($N = 1, \ldots, M$) characterizes an irreversible process of deformation. Such strains may depend upon time as the strains of a viscous material or may be time independent, as the strains of plasticity. In such irreversible processes the free energy must depend upon these additional variables:†

$$F = F(\gamma_{ij}, \gamma_{ij}^N, T). \tag{16}$$

The rate of dissipation is work done by nonconservative stresses s_N^{ij} upon the inelastic strain rates:

$$\dot{D} = s_N^{ij}\dot{\gamma}_{ij}^N \quad \text{(repeated majuscules signify summation)} \tag{17}$$

In accordance with (10),

$$\dot{U} = \dot{F} + S\dot{T} + T\dot{S}.$$

Then eqn (5a) takes the form:

$$\iiint_v [s^{ij}\dot{\gamma}_{ij} - S\dot{T} - \dot{F} - s_N{}^{ij}\dot{\gamma}_{ij}^N]\,dv = 0. \tag{18a}$$

In view of (16), eqn (5b) assumes the form:

$$s^{ij}\dot{\gamma}_{ij} - S\dot{T} - \frac{\partial F}{\partial \gamma_{ij}}\dot{\gamma}_{ij} - \frac{\partial F}{\partial \gamma_{ij}^N}\dot{\gamma}_{ij}^N - \frac{\partial F}{\partial T}\dot{T} = s_N{}^{ij}\dot{\gamma}_{ij}^N \geq 0. \tag{18b}$$

Now presumably the variables γ_{ij}, γ_{ij}^N, T are independent (Subsequently, we display a mechanical

†In general, the free energy is a functional, rather than a function of these variables.

model in which these strains are independently variable). Then it follows that

$$s^{ij} = \frac{\partial F}{\partial \gamma_{ij}}, \qquad S = -\frac{\partial F}{\partial T}; \qquad s_N{}^{ij} = -\frac{\partial F}{\partial \gamma_{ij}^N}. \tag{19a, b, c}$$

In view of (19a, b) the inequality (18) reduces to the form

$$-\frac{\partial F}{\partial \gamma_{ij}^N}\dot{\gamma}_{ij}^N = s_N{}^{ij}\dot{\gamma}_{ij}^N = \dot{D} \geq 0. \tag{20}$$

According to the argument of Valanis[5], $\dot{\gamma}_{ij}^N$ must be related to γ_{ij}, γ_{ij}^N, T; otherwise the inequality (20) could be violated. It follows too that the dissipation rate $\dot{D}$ may be expressed in terms of the strains γ_{ij}, inelastic strain rates $\dot{\gamma}_{ij}^N$ and the temperature T.

Our inelastic strains γ_{ij}^N play the same role as the internal variables of Onsager[6], Meixner[7] and Biot[8]; such continuous variables provide a means to approximate physical alterations of the microstructure, for example, the dislocations in crystalline materials[9, 10].

As a rate $\dot{\gamma}_{ij}^N$ provides a macroscopic measure of an inelastic deformation, a strain rate $\dot{\gamma}_{ij}^E$ $(E = 1, \ldots, L)$ provides a macroscopic measure of elastic deformations. We associate conservative stress components $s_E{}^{ij}$ with the elastic strain rate $\dot{\gamma}_{ij}^E$ and assume that the internal power can be decomposed into conserved and dissipated power:

$$s^{ij}\dot{\gamma}_{ij} = s_E{}^{ij}\dot{\gamma}_{ij}^E + s_N{}^{ij}\dot{\gamma}_{ij}^N. \tag{21}$$

Additionally, we assume that the internal energy depends only upon the elastic strains and the entropy in the manner of (7):

$$U = (\gamma_{ij}^E, S)(E = 1, \ldots, L).$$

Next we introduce the Gibbs potential:

$$G(s_E^{ij}, T) = U - ST - s_E^{ij}\gamma_{ij}^E. \tag{22}$$

Introducing (17), (21) and (22) into (5a) and (5b), we obtain

$$\iiint_v [\gamma_{ij}^E\dot{s}_E^{ij} + S\dot{T} + \dot{G}]\, \mathrm{d}v = 0 \tag{23a}$$

$$\left(\gamma_{ij}^E + \frac{\partial G}{\partial \gamma_{ij}^E}\right)\dot{s}_E{}^{ij} + \left(S + \frac{\partial G}{\partial T}\right)\dot{T} = 0 \tag{23b}$$

and

$$s^{ij}\dot{\gamma}_{ij} - s_E{}^{ij}\dot{\gamma}_{ij}^E = s_N{}^{ij}\dot{\gamma}_{ij}^N \geq 0. \tag{24}$$

Now, presumably the variables $s_E{}^{ij}$ and S are independent; then it follows that

$$\gamma_{ij}^E = -\frac{\partial G}{\partial s_E{}^{ij}}, \qquad S = -\frac{\partial G}{\partial T}. \tag{25a, b}$$

To determine the inelastic strains, we need additional "response functions" as suggested by Kratochvil and Dillon[9].

Variational statements

In keeping with the energy principles embodied in (18a), we define the Gateaux variation:

$$\delta W \equiv \int_{t_0}^{t_1} \iiint_v \Big\{ s^{ij}\delta\gamma_{ij} - S\delta T - \delta F - s_N{}^{ij}\delta\gamma_{ij}^N + \left[\gamma_{ij} - \frac{1}{2}(\mathbf{G}_i\cdot\mathbf{V}_{,j} + \mathbf{G}_j\cdot\mathbf{V}_{,i} - \mathbf{V}_{,i}\cdot\mathbf{V}_{,j})\right)\delta s^{ij} - \left[\frac{1}{\sqrt{g}}(s^{ij}\mathbf{G}_j\sqrt{g})_{,i} + \rho\mathbf{f}\right]\cdot\delta\mathbf{V}\Big\}\,\mathrm{d}v\,\mathrm{d}t + \int_{t_0}^{t_1}\iint_{s_1}[\underline{\mathbf{t}} - s^{ij}n_i\mathbf{G}_j]\cdot\mathbf{V}\,\mathrm{d}s\,\mathrm{d}t + \int_{t_0}^{t_1}\iint_{s_2}[\underline{\mathbf{V}} - \mathbf{V}]\cdot\delta\mathbf{t}\,\mathrm{d}s\,\mathrm{d}t. \tag{26}$$

The variation vanishes,

$$\delta W = 0, \tag{27}$$

for arbitrary $\delta\gamma_{ij}$, δT, $\delta\gamma_{ij}^N$, δs^{ij}, $\delta\mathbf{V}$, if and only if (1) the constitutive eqns (19) and (20) are satisfied, (2) the kinematic equations are satisfied in v (the strain-displacement relations) and on s_2 and (3) the dynamic equations are satisfied in v (differential equations of motion) and on s_1.

The stationary condition (27) is a version of Hamilton's principle. If the behavior is time-independent, isothermal and elastic (mechanically conservative), then a potential W exists and (27) expresses the theorem of Hu–Washizu[11, 12].

In accordance with the energy principle (23a), we define a variation:

$$\delta I \equiv \int_{t_0}^{t_1}\iiint_v \Big\{\gamma_{ij}^E\delta s_E{}^{ij} + S\delta T + \delta G - \left[\gamma_{ij} - \frac{1}{2}(\mathbf{G}_i\cdot\mathbf{V}_{,j} + \mathbf{G}_j\cdot\mathbf{V}_{,i} - \mathbf{V}_{,i}\cdot\mathbf{V}_{,j})\right]\delta s^{ij} - \left[\frac{1}{\sqrt{g}}(s^{ij}\mathbf{G}_j\sqrt{g})_{,i} + \rho\mathbf{f}\right]\cdot\delta\mathbf{V}\Big\}\,\mathrm{d}v\,\mathrm{d}t + \int_{t_0}^{t_1}\iint_{s_1}[\underline{\mathbf{t}} = s^{ij}n^i\mathbf{G}_j]\cdot\delta\mathbf{V}\,\mathrm{d}s + \int_{t_0}^{t_1}\iint_{s_2}[\underline{\mathbf{V}} - \mathbf{V}]\cdot\delta\mathbf{t}\,\mathrm{d}s \tag{28}$$

The variation vanishes,

$$\delta I = 0, \tag{29}$$

for arbitrary δs_{ij}^E, δT, δs^{ij}, $\delta\mathbf{V}$, if and only if, (1) the constitutive eqns (25) are satisfied, (2) the kinematic equations are satisfied in v (the strain-displacement relations) and on s_2, and (3) the dynamic equations are satisfied in v (differential equations of motion) and on s_1.

If the behavior is time-independent, is isothermal and elastic, then a potential I exists and (29) is the stationary theorem of Hellinger–Reissner[13, 14].

INCREMENTAL FORMULATION

In a nonconservative problem of mechanics, one must anticipate a solution by incremental steps, that is, by solving a succession of linear equations. Consequently, the practical formulation is a system of linear equations which govern the incremental stresses, strains and temperatures. To achieve this end, we assume that the thermal and dynamic conditions are satisfied in a reference state which we signify by a bar (—), and assume further that the functions F and D are analytic in a neighborhood of the reference state. The relevant terms are linear and quadratic in the increments of the dependent variables.

CLASSICAL PLASTICITY

The mechanical model of Fig. 1a displays certain basic features of classical plasticity. The strain of the model (γ_{ij}) is the sum of the strains of the elastic element (γ_{ij}^E) and the plastic element (γ_{ij}^P):

$$\acute{\gamma}_{ij} = \acute{\gamma}_{ij}^E + \acute{\gamma}_{ij}^P \tag{30}$$

where the prime (′) signifies an increment. The stress upon the model (s^{ij}) is the stress upon the elastic element ($s_E{}^{ij}$) and the plastic element ($s_P{}^{ij}$):

$$s_E{}^{ij} = s_P{}^{ij} = s^{ij}.$$

In the classical theory, yielding occurs, if and only if, the prevailing stress ($\bar{s}^{ij}$) satisfies a yield condition,

$$f(\bar{s}^{ij}) = 0, \tag{31}$$

and the stress increments satisfy a loading condition:

$$\frac{\partial f}{\partial s^{ij}} \acute{s}^{ij} = G_P \acute{\lambda} \geq 0, \tag{32}$$

where $\lambda' > 0$, $G_P > 0$ if the material hardens and $G_P = 0$ if the material is ideally plastic. Additionally, the classical theory asserts that

$$\dot{D} = s^{ij} \dot{\gamma}_{ij}^{P} \tag{33}$$

which leads to the normality condition

$$\acute{\gamma}_{ij}^{P} = \acute{\lambda} \frac{\partial f}{\partial s^{ij}}. \tag{34}$$

In addition, we have the elastic strain increment:

$$\acute{\gamma}_{ij}^{E} = \bar{D}_{ijkl} \acute{s}^{kl} \tag{35}$$

where $\bar{D}_{ijkl}$ is a component of the prevailing flexibility tensor, the inverse of the stiffness tensor (E^{ijkl}). By means of (30), (32), (34) and (35), the stress-strain equation takes the form:

$$\acute{s}^{ij} = E_T{}^{ijkl} \acute{\gamma}_{kl} \tag{36}$$

where $E_T{}^{ijkl}$ denotes a so-called tangent modulus:

$$E_T{}^{ijkl} = E^{ijkl} - \frac{E^{ijmn} E^{klpq} \dfrac{\partial f}{\partial s^{mn}} \dfrac{\partial f}{\partial s^{pq}}}{G_P + E^{ijkl} \dfrac{\partial f}{\partial s^{ij}} \dfrac{\partial f}{\partial s^{kl}}}. \tag{37}$$

We note that the classical theory is expressed by constitutive equations which are *linear* in the increments of stress and strain. The constitutive equations can be derived according to the energy principles (18) and (23), or the variational conditions (27) and (29). The linear equations are obtained from functions, or functionals, which are *quadratic* in the *increments* of their respective arguments. For example, in accordance with (32), (33) and (34), the dissipation function is expanded in the form:

$$D = \bar{D} + \bar{s}^{ij} \acute{\gamma}_{ij}^{P} + \frac{1}{2} \acute{s}^{ij} \acute{\gamma}_{ij}^{P} + \cdots = \bar{D} + \bar{s}^{ij} \frac{\partial f}{\partial s^{ij}} \acute{\lambda} + \frac{1}{2} G_P (\acute{\lambda})^2 + \cdots.$$

Quadratic forms of the functionals W and I are given in a previous work[15]: Then the stationary conditions of (27) and (29) provide the equations governing increments; the former ($\delta W = 0$) provides incremental stress-strain relations and the latter ($\delta I = 0$) provides incremental strain-stress relations.

PLASTICITY WITHOUT A YIELD CONDITION

The yield condition of classical plasticity signals an abrupt transition from the conditions of elasticity to inelasticity. Consequently, portions of the body behave elastically while adjoining portions behave inelastically; the elastic behavior is governed by the constitutive equations of elasticity and the inelastic behavior by the equations of the inelasticity. Moreover, the interface moves as yielding progresses. In practice, different computational procedures are needed for the elastic and inelastic parts; additionally, a procedure is needed to follow the movement of the interface as loading progresses.

Plasticity without a yield condition admits a gradual evolution of inelastic strain from the onset of loading. Valanis' endochronic theory[2] holds much promise as a basis for practical computations of elastic-plastic shells.

RATE INDEPENDENT BEHAVIOR

If the material is insensitive to the rates of deformation and/or loading, then time does not appear explicitly. Indeed, the time t in (30) through (39) is any variable which increases monotonically with changes of state.

In his book[16], Hill introduced a measure of plastic deformation, the second invariant of the plastic strain increment:

$$(\mathrm{d}\gamma^P)^2 \equiv \mathrm{d}\gamma^P_{ij}\,\mathrm{d}\gamma_P{}^{ij}. \tag{38}$$

The invariant $\mathrm{d}\gamma^P$ is the increment of arc length in the space of plastic strain. According to the Mises[17] yield condition and the associated flow law of Prandtl–Reuss[18, 19], the invariant $(\mathrm{d}\gamma^P)^2$ is proportional to the dissipation rate and, therefore, increases monotonically during any plastic deformation.

In subsequent works on inelastic materials, Rivlin and Pipkin[20] introduced the second invariant of the strain increment:

$$(\mathrm{d}\gamma)^2 = \mathrm{d}\gamma_{ij}\,\mathrm{d}\gamma^{ij}. \tag{39}$$

Here, γ is the arc length in strain space, but called "time" since it measures the events which alter the material. A generalization was given by Valanis[2], who defined an "intrinsic time measure" ζ; in the case of rate-independent behavior,

$$(\mathrm{d}\zeta)^2 \equiv a^2 p^{ijkl}\,\mathrm{d}\gamma_{ij}\,\mathrm{d}\gamma_{kl}. \tag{40}$$

In addition, Valanis introduced an "intrinsic time scale" $z = z(\zeta)$, such that

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} > 0 \; (0 < \zeta < \infty). \tag{41}$$

The latter offers further possibilities to fit the stress-strain relation of a given material. The "time" z reduces to γ if

$$a^2 p^{ijkl} = G^{ik}G^{jl}, \qquad z = \zeta \tag{42a, b}$$

where G^{ij} is a component of the covariant metric tensor for the coordinates of the deformed body.

VALANIS' ENDOCHRONIC THEORY OF PLASTICITY

In the case of rate-independent behavior, the dot ($\dot{\ }$) in the preceding equations may signify the derivative with respect to any "intrinsic time" z. With the understanding that the variables (γ_{ij}, γ^N_{ij}, T) are small and characterize a small change from a reference state, Valanis employed a general quadratic form for $F(\gamma_{ij}, \gamma^N_{ij}, T)$ and a quadratic form of the dissipation "rate":

$$\dot{D} = \frac{\mathrm{d}D}{\mathrm{d}z} = b_N{}^{ijkl}\frac{\mathrm{d}\gamma^N_{ij}}{\mathrm{d}z}\frac{\mathrm{d}\gamma^N_{kl}}{\mathrm{d}z}. \tag{43}$$

The eqns (19a, c) are then linear differential equations, analogous to the equations governing linear viscoelastic materials wherein real time t is replaced by an intrinsic time z. Accordingly one obtains stress-strain relations in the form of hereditary integrals. For example, in an isotropic elastic-plastic material the deviators of stress and strain may be related as follows:

$$\tilde{s}_{ij} = 2\int_{z0}^{z} G(z-\tau)\frac{\partial\tilde{\gamma}_{ij}}{\partial\tau} \tag{44}$$

where

$$2G(z) = 2G_1 + 2G_0 e^{-\alpha z}. \tag{45}$$

More recently Bhandari and Oden[3] also utilized (40) and (44) with

$$G(z) = G_0\, e^{-\alpha z}. \tag{46}$$

In applications of the foregoing theory, Valanis[2] and Bhandari[3], utilized, respectively:

$$\frac{d\zeta}{dz} = 1 + \beta\zeta, \qquad \frac{d\zeta}{dz} = p \tag{47a, b}$$

where β and p are positive constants.

Valanis arrived at the form (44) by integrating his linear equations governing the small strain of a rate-independent material. He observed too that the form (44) implies an incremental relation which has the appearance of the Prandtl–Reuss relation:

$$d\tilde{\gamma}_{ij} = \frac{\alpha\, dz}{2G_0}\tilde{s}_{ij} + \frac{1}{2G_0}\, d\tilde{s}_{ij}. \tag{48}$$

Bhandari and Oden[3] indicate that the form (44) was suggested by previous work[21] on the behavior of poly-crystalline graphite. It is quite likely that these investigators were also motivated by the similarities between the linear equations of the so-called Maxwell material and the equations of the Reuss material as well as the similarities between the stress-time and stress-strain relations.

In the previous works[2, 3] the authors employed the relation (46), with their respective forms z (ζ), for the initial loading, but each introduced a different form to accomodate unloading. Valanis employed the form (45). Bhandari and Oden altered the exponential by a factor such that

$$G(z) = G_0\, e^{k\hat{z}}\, e^{-\alpha(z)}.$$

Here $\hat{z}$ is the value z at the onset of unloading and k is a parameter dependent upon material parameters and the number of unloading cycles. Both of the foregoing theories were shown to describe the stress-strain relations for certain materials. Both provide a means to describe cyclic loading. Additionally, Valanis' theory was shown to describe so-called"cross-hardening", (the effect of shear prestrain upon the tensile stress-strain-relation), whereas, a linear relation ($z = z_0 + k\zeta$), like (47b) does not accomodate the cross-hardening.

Our immediate concern is the physical interpretation and implications of the foregoing theories. Some insight is provided by certain mechanical models which lead to the forms (44) and (45).

ELASTIC–PLASTIC MODEL "1"

The simple mechanical model of Fig. 1a consists of a spring and slider mechanism. The elongation of the former is analogous to an elastic strain γ^E and the latter a plastic strain γ^P. The force upon the assembly is analogous to the stress s. The slider exhibits a frictional resistance and dissipates energy; it may require finite force to initiate the elongation, as the yield condition of classical plasticity. However, to develop a simple endochronic theory, we suppose that the

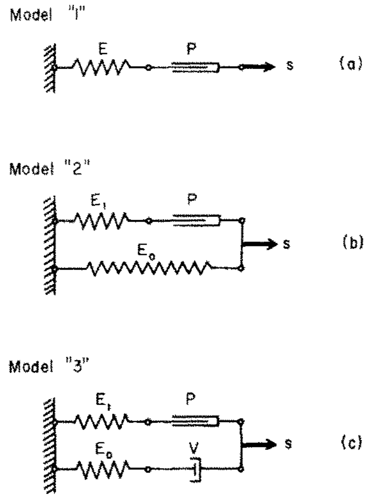

Fig. 1.

incremental strain of the slider is proportional to the stress:

$$d\gamma^P = \frac{dz}{\lambda} s, \qquad \left(\frac{dz}{\lambda} > 0\right).$$

The dissipation rate follows:

$$\dot{D} = s\dot{\gamma}^P = \lambda \frac{d\gamma^P}{dz} \dot{\gamma}^P. \tag{49}$$

We suppose that the free energy is the work expended upon the elastic element; to quadratic terms in the strain increment ($\acute{\gamma}$) and plastic strain increment ($\acute{\gamma}^P$), we have

$$F(\acute{\gamma} - \acute{\gamma}^P) = \bar{F} + \bar{s}(\acute{\gamma} - \acute{\gamma}^P) + \frac{E}{2}(\acute{\gamma} - \acute{\gamma}^P)^2. \tag{50}$$

In accordance with (19a, c) and (20),

$$s = \bar{s} + E(\acute{\gamma} - \acute{\gamma}^P), \qquad s_P = \bar{s} + E(\acute{\gamma} - \acute{\gamma}^P) = \lambda \frac{d\gamma^P}{dz}.$$

Eliminating s_p and γ^P we obtain

$$\frac{ds}{dz} + \frac{E}{\lambda} s = E \frac{d\gamma}{dz}. \tag{51}$$

Conceptually, the variable z must be monotonically increasing with any measure of plastic deformation. The solution follows:

$$s = \int_{z_0}^{z} E\, e^{-(E/\lambda)(z-\tau)}\, d\gamma(\tau). \tag{52}$$

In Valanis' Theory, z is the "intrinsic time". He has proposed a particularly simple relation between the "intrinsic time z" and the "strain measure" ζ, specifically,

$$\frac{d\zeta}{dz} = (1 + \beta\zeta) \tag{53}$$

If (53) is employed in (52), then continuous straining ($d\zeta = d\gamma$), from the initial unstressed state $s(\zeta_0) = 0$ produces the stress:

$$s = \frac{E}{\alpha + \beta}(1 + \beta\zeta)\left[1 - \left(\frac{1 + \beta\zeta_0}{1 + \beta\zeta}\right)^{1+(\alpha/\beta)}\right], \qquad \alpha \equiv \frac{E}{\lambda}. \tag{54a}$$

Unstraining ($d\zeta = -d\gamma$) from the stressed state $s(\zeta_1) = s_1$ produces the stress

$$s = \frac{E(1 + \beta\zeta)}{\alpha + \beta}\left[-\left(\frac{1 + \beta\zeta_0}{1 + \beta\zeta}\right)^{1+(\alpha/\beta)} - 1 + 2\left(\frac{1 + \beta\zeta_1}{1 + \beta\zeta}\right)^{1+(\alpha/\beta)}\right]. \tag{54b}$$

Upon initial loading and at the onset of reloading from $s = 0$, $ds/d\gamma = E$. During monotonic straining, the stress-strain relation approaches the asymptote:

$$s = \frac{E}{\alpha + \beta}(1 + \beta\gamma)$$

As noted by Valanis, the parameter $\beta > 0$ characterizes hardening.

ELASTIC–PLASTIC MODEL "2"

A basic, yet questionable, assumption of classical plasticity is the additivity of elastic and plastic strain increments and the related notion that so-called plastic work ($s^{ij}\dot{\gamma}^P_{ij}$) is entirely dissipated. According to Lehmann[22], the assumption is contradicted by experimental results[23] on polycrystalline materials. In effect, a part of the stress is conservative, so that

$$s^{ij}\dot{\gamma}^P_{ij} \neq \dot{D}$$

Our simple model "2" of Fig. lb. serves to illustrate the point.

Now, the free energy function includes the energy of the elastic element "0", as well as the energy of the element "1":

$$F = \bar{F} + \bar{s}_0\acute{\gamma} + \bar{s}_1(\acute{\gamma} - \acute{\gamma}^P) + \frac{1}{2}E_0\acute{\gamma}^2 + \frac{1}{2}E_1(\acute{\gamma} - \acute{\gamma}^P)^2 \tag{22}$$

Again, the dissipation rate may take the form of (49):

$$\dot{D} = \lambda \frac{d\gamma^P}{dz}\dot{\gamma}^P \tag{56}$$

In accordance with (19) and (20),

$$s = \bar{s}_0 + \bar{s}_1 + E_0\acute{\gamma} + E_1(\acute{\gamma} - \acute{\gamma}^P)$$

$$\lambda\frac{d\gamma^P}{dz} = \bar{s}_1 + E_1(\acute{\gamma} - \acute{\gamma}^P)$$

Eliminating γ^P, we obtain

$$\frac{ds}{dz} + \frac{E_1}{\lambda}s = (E_0 + E_1)\frac{d\gamma}{dz} + \frac{E_1E_0}{\lambda}\gamma \tag{57}$$

Again the solution is given in the form of the hereditary integral

$$s = E_0\gamma + E_1\int_{z_0}^{z} e^{-\alpha(z-\tau)}\,d\gamma(\tau), \qquad \alpha = \frac{E_1}{\lambda} \tag{58}$$

According to Valanis[2, 24] and Morrow[25], the stress-strain relation (58) provides a better discription of metals under cyclic loading than (52)

VISCOUS-ELASTIC-PLASTIC MODEL "3"

Our mechanical models of Valanis' theory offer some physical interpretations, but also provide a mechanism to develop constitutive equations which possess specific attributes. In particular, creep and relaxation are introduced by the addition of viscous elements. The model "3" of Fig. 1c is a simple example which contains one viscous element "V". The dissipation is now rate dependent; in place of (56), we have

$$\dot{D} = \lambda \frac{d\gamma^P}{dz} \dot{\gamma}^P + \mu \frac{d\gamma^V}{dt} \dot{\gamma}^V \tag{59}$$

The stress-strain relation assumes the form:

$$s = E_1 \int_0^t e^{-(E_1/\lambda)(z-\acute{z})} \frac{d\gamma}{d\acute{z}} \frac{d\acute{z}}{d\tau} d\tau + E_0 \int_0^t e^{-(E_0/\mu)(t-\tau)} \frac{d\gamma}{d\tau} d\tau. \tag{60}$$

Here, we presume an initial unstressed state and continuity of the strain history. Also, we note that the intrinsic time z is an implicit function of real time t.

CONSTITUTIVE EQUATIONS OF ISOTROPIC ENDOCHRONIC THEORY

The equations of the models "1", "2" and "3" are readily extended to three-dimensions. The counterparts of the free energy (55) and dissipation (59) assume the following forms:

$$F = \bar{F} + \bar{s}_0^{ij}(\acute{\gamma}_{ij} - \acute{\gamma}_{ij}^V) + \bar{s}_1^{ij}(\acute{\gamma}_{ij} - \acute{\gamma}_{ij}^P) + \frac{1}{2} E_0^{ijkl}(\acute{\gamma}_{ij} - \acute{\gamma}_{ij}^V)(\acute{\gamma}_{kl} - \acute{\gamma}_{kl}^V) + \frac{1}{2} E_1^{ijkl}(\acute{\gamma}_{ij} - \acute{\gamma}_{ij}^P)(\acute{\gamma}_{kl} - \acute{\gamma}_{kl}^P) \tag{61a}$$

$$\dot{D} = \lambda^{ijkl} \frac{d\gamma_{ij}^P}{dz} \dot{\gamma}_{kl}^P + \mu^{ijkl} \frac{d\gamma_{ij}^V}{dt} \dot{\gamma}_{kl}^V. \tag{61b}$$

If the material is isotropic then the moduli, E_0^{ijkl} and E_1^{ijkl}, the plasticity coefficients λ^{ijkl} and viscosity coefficients μ^{ijkl} are each expressed in terms of two coefficients:

$$E_0^{ijkl} = A_1 G^{ij} G^{kl} + A_2 G^{ik} G^{jl} \tag{62a}$$

$$E_1^{ijkl} = B_1 G^{ij} G^{kl} + B_2 G^{ik} G^{jl} \tag{62b}$$

$$\mu^{ijkl} = \mu_1 G^{ij} G^{kl} + \mu_2 G^{ik} G^{jl} \tag{62c}$$

$$\lambda^{ijkl} = \lambda_1 G^{ij} G^{kl} + \lambda_2 G^{ik} G^{jl} \tag{62d}$$

here G^{ij} denotes a component of the contravariant metric tensor of the convected coordinate system (δ^{ij} in a Cartesian system). Now, the governing equations are uncoupled differential equations which relate the corresponding components of the stress and strain deviations and an equation which relates the first invariants. In place of (60) we obtain

$$\tilde{s}_{ij} = A_2 \int_{t_0}^t e^{-\alpha_0(t-\tau)} d\tilde{\gamma}_{ij} + B_2 \int_{t_0}^t e^{-\alpha_1(z-\acute{z})} d\tilde{\gamma}_{ij} \tag{63a}$$

$$\frac{1}{3} s_i^i = \left(A_1 + \frac{1}{3} A_2\right) \int_{t_0}^t e^{-\rho_0(t-\tau)} d\gamma_i^i + \left(B_1 + \frac{1}{3} B_2\right) \int_{t_0}^t e^{-\rho_1(z-\acute{z})} d\gamma_i^i \tag{63b}$$

where

$$\alpha_0 = \frac{A_2}{\mu_2}, \qquad \alpha_1 = \frac{B_2}{\lambda_2} \tag{64a, b}$$

$$\rho_0 = \frac{3A_1 + A_2}{3\mu_1 + \mu_2}, \qquad \rho_1 = \frac{3B_1 + B_2}{3\lambda_1 + \lambda_2} \tag{64c, d}$$

$$z = z(\zeta), \qquad d\zeta^2 = d\gamma^{ij} d\gamma_{ij}, \qquad \gamma_{ij} = \gamma_{ij}(t).$$

The constitutive equations (63a, b) govern the simpler models "2" and "1" in the limiting cases, $\mu^{ijkl} \to \infty$ and $E_0^{ijkl} \to 0$.

The Valanis' theory applies as well to anisotropic materials, which are governed by coupled equations, and to more elaborate models, which may lead to equations of higher order.

CORRELATION OF ENDOCHRONIC AND CLASSICAL PLASTICITY

The mechanical model of 1 of Fig. 1a may be regarded as a basic model of the endochronic *and* classical theories. In either case, the governing equations take the incremental forms:

$$\acute{\gamma}_{ij} = D_{ijkl}\acute{s}^{kl} + h_{ij}(s^{ij})\acute{z} \tag{65}$$

Comparison is possible if yielding is enforced in accordance with the criterion of the classical theory:

$$h_{ij}\acute{s}^{ij} \equiv G_P \acute{z} \geq 0 \tag{66}$$

With the condition (66), i.e. $G_P > 0$, the eqn (65) may be solved for $\acute{z}$:

$$\acute{z} = \frac{E^{ijmn} h_{mn} \acute{\gamma}_{ij}}{G_P + E^{ijmn} h_{ij} h_{mn}}. \tag{67}$$

The stress-strain relation follows:

$$\acute{s}^{pq} = \left(E^{ijpq} - \frac{E^{ijkl} E^{pqmn} h_{kl} h_{mn}}{G_p + E^{ijmn} h_{ij} h_{mn}} \right) \acute{\gamma}_{ij}. \tag{68}$$

If h_{mn} is taken in the form of eqn (48) and E^{ijkl} is the component of the isotropic tensor, then

$$\acute{z} = \frac{\alpha \tilde{s}^{mn} \acute{\gamma}_{mn}}{G_P + \frac{\alpha^2}{2G_0}(\tilde{s}^{mn}\tilde{s}_{mn})}. \tag{69}$$

If Valanis' form (53) is substituted into the left side of (69), we have

$$G_P = \alpha \tilde{s}^{mn}(1 + \beta\zeta)\frac{d\gamma_{mn}}{d\zeta} - \frac{\alpha^2}{2G_0}(\tilde{s}^{mn}\tilde{s}_{mn}). \tag{70}$$

The stress-strain relation (68) takes the form:

$$\acute{s}^{pq} = \left[E^{pqij} - \frac{\alpha \tilde{s}^{pq} \tilde{s}^{ij}}{\tilde{s}^{mn} \frac{d\gamma_{mn}}{dz}} \right] \acute{\gamma}_{ij} \tag{71a}$$

or with the Valanis' hardening of (53),

$$\acute{s}^{pq} = \left[E^{pqij} - \frac{\alpha \tilde{s}^{pq} \tilde{s}^{ij}}{(1 + \beta\zeta)\tilde{s}^{mn} \frac{d\gamma_{mn}}{d\zeta}} \right] \acute{\gamma}_{ij}. \tag{71b}$$

The incremental form (68) embodies features of the classical plasticity and the endronic theory as well, if the yield condition (66) is eliminated; the form (71a) follows the Prandtl–Reuss theory and (71b) follows the Valanis' theory. The bracketed coefficients of (71) serve as the tangent moduli of (37); however, the former contain the derivatives $d\gamma_{mn}/dz$. Consequently, a corner in the strain path introduces a corresponding discontinuity in the coefficients.

The equations of our model "2" were employed by Valanis to describe cycles of loading and unloading. If the plastic element deforms according to the Prandtl–Reuss theory, then a three-dimensional counterpart of the incremental relation (57) follows:

$$\acute{\tilde{\gamma}}_{ij}^{P} = \frac{\alpha}{2G_1}(\tilde{s}_{ij} - 2G_0\tilde{\gamma}_{ij})\acute{z}. \tag{72a}$$

If the Mises[17] yield condition and Prandtl–Reuss equations are modified to account for kinematic hardening[26, 27], then the incremental plastic strain is given by the equation:

$$\acute{\tilde{\gamma}}_{ij}^{P} = \frac{\alpha}{2G_1}(\tilde{s}_{ij} - \alpha_{ij})\acute{z} \tag{72b}$$

Prager[26] and Ziegler[27] have proposed alternative theories which govern the variable α_{ij}; in either theory, variation occurs only during yielding which obeys a yield condition. Equation (72a) implies yet another alternative wherein the variable α_{ij} is the conservative deviatoric stress component, $\alpha_{ij} = \tilde{s}_0{}^{ij} = 2G_0\tilde{\gamma}_{ij}$. In the spirit of the endochronic theory, α_{ij} varies continuously with any strain, just as yielding occurs continuously with any strain.

In the view of the foregoing similarity, one can expect the model "2" to exhibit a Bauschinger effect, similar to the effect of kinematic hardening and, indeed, Valanis has noted the effect[2].

MODIFIED ENDOCHRONIC THEORY

The endochronic theory is far more general than the foregoing models may suggest: Valanis[24, 28] has discussed various forms of hardening $z(\zeta)$ and Bazant[29, 30] has proposed forms in which the behavior of concrete and sand depend upon invariants of strain *and* stress. However, most metals do exhibit essentially linear and elastic behavior during unloading.

In his original version, Valanis did not distinguish unloading. Since we anticipate step-wise linear computations, we propose to distinguish unloading by the condition

$$\acute{s}_P{}^{ij}\acute{\gamma}_{ij}^{P} < 0. \tag{73}$$

Now, the stress-strain equation, e.g. (71) can be replaced by the equation of elasticity, linear or nonlinear, if (73) holds. In practice, the advantages of the endochronic theory are retained. The modified theory may also exhibit a type of kinematic hardening, as the condition (73) applies only to the dissipative stress (or stresses), e.g. s_P of model "2".

CONCLUSION

In practice, the nonlinear and nonconservative problem are treated by incremental, step-wise linear, computations. Then, one can employ (68), in which G_P may depend upon strain and/or stress invariants; $z(\zeta)$ may assume any form as dictated by experiments. However, to illustrate certain features we employ Valanis' form (53) and the consequent stress-strain eqn (71). Then the properties of the materials, which follow the models 1, 2 and 3, are embodied in four parameters:

$$\alpha = \frac{E_1}{\lambda}, \qquad \beta, \qquad \varphi = \frac{E_0}{E_1}, \qquad \eta = \frac{E_1}{\mu}.$$

In his earlier work[2], Valanis showed the excellent agreement between his theory and experiments on copper[31]. A plot of the simple stress-strain relation is displayed in Fig. 2, which shows a consequence of our modification, as described in the preceeding section. In addition, Valanis showed agreement with experiments on copper under cyclic loading[32, 33]. The latter experiments revealed the Bauschinger effect which calls for a hardening of the type exhibited by

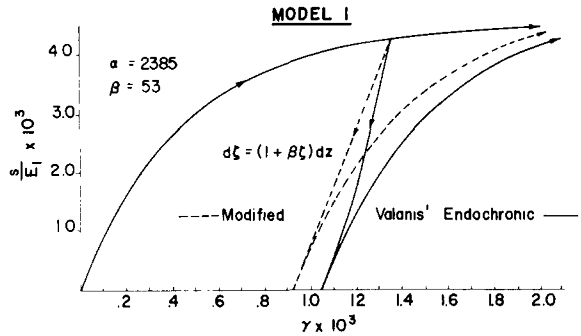

Fig. 2.

our model 2. Again, the endochronic theory of Valanis and the modified version provide the curves of Fig. 3. The influence of the additional elastic element is displayed more vividly in Fig. 4: Here, the initial loading curves of material $1(\varphi = 0)$ and $2(\varphi = 0.100)$ are similar; one approaches the other asymptotically. However, the response to cyclic loading is vastly different; the former exhibits a pronounced Bauschinger effect and a large hysteresis loop while the latter exhibits a slight hardening and a much smaller loop. The effect can be explained via the model $2(\varphi = 0.100)$ of our modified theory: Upon unloading elastically to point A, the *dissipative* stress upon the plastic element is reversed and plastic flow ensues. Again, upon reloading to point B the *dissipative* stress changes sign and again plastic deformation ensues. It is noteworthy, that the modified theory retains the essential features of the endochronic theory, yet the former admits linear elastic unloading, so characteristic of most structural metals.

The modified theory, provides a close correlation with experiments upon an aluminum alloy under cyclic loading, as depicted in Fig. 5[34]. As a material hardens ($\beta > 0$) under cyclic loading,

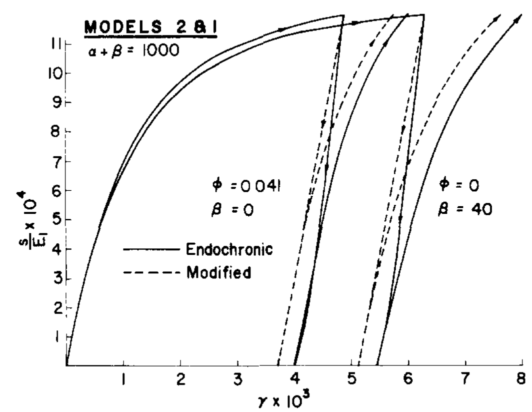

Fig. 3.

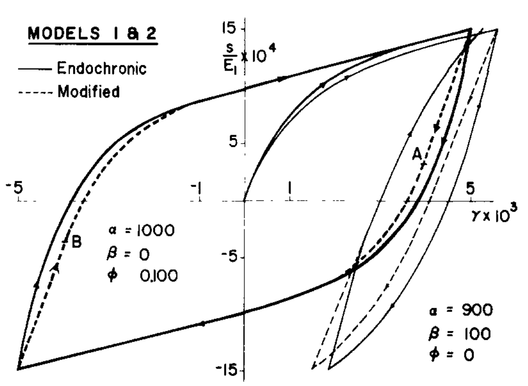

Fig. 4.

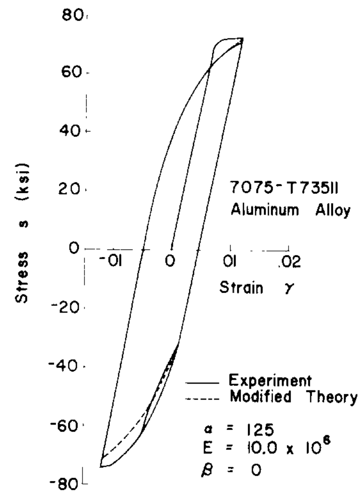

Fig. 5.

progressively less inelastic strain and less dissipation accompanies subsequent cycles; the response approaches the elastic.

An effect of the viscosity in model 3 is shown in Fig. 6. Under rapid loading the viscosity is not evident ($\eta \to 0$) and the material is correspondingly harder; under slow loading this hardness is not evident ($\eta \to \infty$).

It is interesting, but not surprising, that a theory, which describes the mechanical attributes of a solid, is analogous to a mechanical assembly, for the micro-structure of the solid is likewise an assembly of crystals, molecules, impurities, defects, dislocations, etc., which can distort elastically, slip plastically, or otherwise move or grow to impede or promote subsequent deformations.

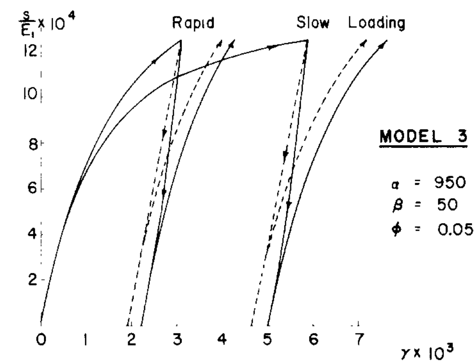

Fig. 6.

The computations of the foregoing examples were executed by the incremental, piece-wise linear, procedure. The same logic can be utilized to effect the simultaneous solution of the constitutive, kinematic and dynamic equations of a structural member. The elastic and viscous elements of our models and the consequent equations need not be linear; indeed, the incremental procedure admits most forms of material and geometrical nonlinearities.

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